On the tree-width of even-hole-free graphs

Dewi Sintiari

LIP, ENS de Lyon

Joint work with Pierre Aboulker, Isolde Adler, Eun Jung Kim, Nicolas Trotignon

European Congress of Mathematics, Portorož, Slovenia

June 23, 2021

Even-hole-free graphs

- H is an induced subgraph of G if H can be obtained from G by *deleting vertices*

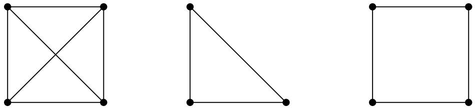


Figure: A graph, an induced subgraph, and a non-induced subgraph

- G is H-free if no induced subgraph of G is isomorphic to H
- When $\mathcal{F}$ is a family of graphs, $\mathcal{F}$-free means H-free, $\forall H \in \mathcal{F}$

Even-hole-free graphs

- Even hole: induced cycle of even length (i.e. no chord in the cycle)
- G is even-hole-free means G does not *contain* an even hole
  - Even-hole-free: chordal graphs, complete graphs
  - Not even-hole-free:

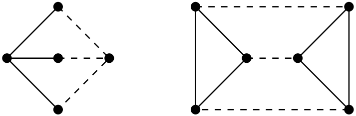


Figure: Theta and prism

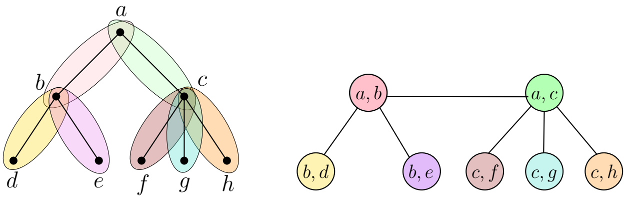
Motivation

- Relation to *perfect graphs* (G is perfect if for every induced subgraph H of G, $\chi(H) = \omega(H)$)

	EHF graphs	Perfect graphs
Basic graphs	cliques, holes long pyramids, nontrivial basic	bipartite, $\overline{\text{bipartite}}$, L(bipartite), $\overline{\text{L(bipartite)}}$ doubled graphs
Cutsets	2-join, star cutset	clique cutset, 2-join, $\overline{\text{2-join}}$ homogeneous pair, balanced skew partition
Polynomial α, χ	?	YES

Tree-width (*intuitively*)

Tree decomposition



- Tree decomposition of G: "gluing" the pieces of subgraphs of G in a tree-like fashion
  - width of T = the size of the largest bag - 1
  - tree-width of G: the minimum over the width of tree decomposition of G

Algorithmic use of tree-width

Theorem (Courcelle, 1990)

Every graph property definable in the monadic second-order logic (MSO) formulas can be decided in linear time on class of graphs of bounded tree-width.

Some graph problems expressible in MSO:

- maximum independent set, maximum clique, coloring

Which even-hole-free graphs have *bounded* tree-width?

Observation: *The tree-width of the class is unbounded*

- *Planar* ehf: $tw \leq 49$ [Silva, da Silva, Sales, 2010]
- *Pan-free* ehf: $tw \leq 1.5\omega(G) - 1$ [Cameron, Chaplick, Hoàng, 2015]
- K_3*-free* ehf: $tw \leq 5$ [Cameron, da Silva, Huang, Vušković, 2018]
- *Cap-free* ehf: $tw \leq 6\omega(G) - 1$ [Cameron, da Silva, Huang, Vušković, 2018]

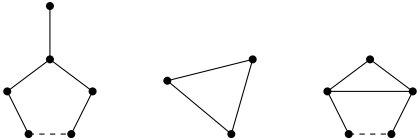


Figure: Pan, triangle, and cap

Even-hole-free graphs with *unbounded* tree-width

- Diamond-free ehf has unbounded *rank-width* [Adler, Le, Müller, Radovanović, Trotignon, Vušković, 2017]

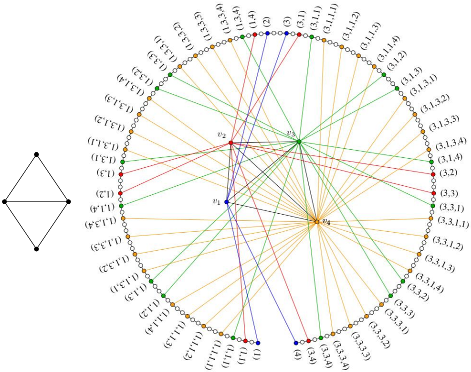


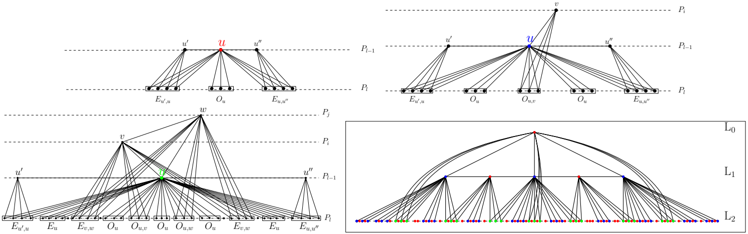
Figure: A diamond-free ehf graph with large rank-width

Even-hole-free graphs with *unbounded* tree-width

Do even-hole-free graphs of bounded ω have bounded tw?
[Cameron, Chaplick, Hoàng, 2018]

- No, because K_4-free even-hole-free graphs have unbounded tree-width [S., Trotignon, 2019]

Even-hole-free graphs with *unbounded* tree-width



- The graphs have large degree and contain large clique minor
  clique minor: pairwise adjacent connected subgraphs

Question: **Are these two conditions necessary?**

Main questions

- What is the tree-width of even-hole-free graphs with no clique minor?

- What is the tree-width of even-hole-free graphs with bounded maximum degree?

Main questions

- What is the tree-width of even-hole-free graphs with no clique minor?
    - It is bounded [Aboulker, Adler, Kim, S., Trotignon, 2020]
- What is the tree-width of even-hole-free graphs with bounded maximum degree?
    - We prove partial results (for $\Delta = 3$ and a subclass of ehf graphs with $\Delta = 4$)
    - It is bounded [Abrishami, Chudnovsky, Vušković, 2020]

1st part: even-hole-free graphs with no H-minor

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

(Theta, prism)-free graphs with no H-minor for some graph H have bounded tree-width.

1st part: even-hole-free graphs with no H-minor

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

(Theta, prism)-free graphs with no H-minor for some graph H have bounded tree-width.

Theorem (induced-wall theorem for H-minor-free graph)

$\forall H$, *if* G *is* H*-minor-free with* $tw(G) \geq f_H(k)$, *then* G *contains a* $(k \times k)$*-wall (possibly subdivided) or the line graph of a chordless* $(k \times k)$*-wall (or call it co-wall) as an induced subgraph.*

Even-hole-free graphs with no H-minor

Theorem (induced-wall theorem for H-minor-free graph)

$\forall H$, if G is H-minor-free with $tw(G) \geq f_H(k)$, then G contains a $(k \times k)$-wall (possibly subdivided) or the line graph of a chordless $(k \times k)$-wall (or call it co-wall) as an induced subgraph.

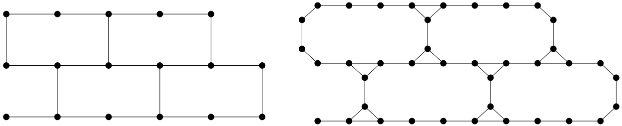


Figure: A (3×3)-wall and the (3×3)-co-wall

Even-hole-free graphs with no H-minor

Theorem (induced-wall theorem for H-minor-free graph)

$\forall H$, if G is H-minor-free with $tw(G) \geq f_H(k)$, then G contains a $(k \times k)$-wall (possibly subdivided) or the line graph of a chordless $(k \times k)$-wall (or call it co-wall) as an induced subgraph.

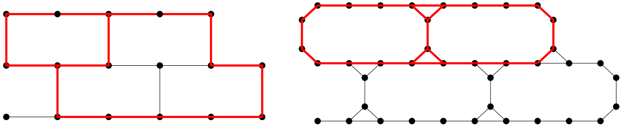


Figure: A (3×3)-wall and the (3×3)-co-wall

Even-hole-free graphs with no H-minor

Theorem (Fomin, Golovach, Thilikos, 2011)

For every H and an integer k, there exists a function $f_H(k)$ s.t. for every connected H-minor free graph G with $tw(G) \geq f_H(k)$, G contains either Γ_k or Π_k as a contraction.

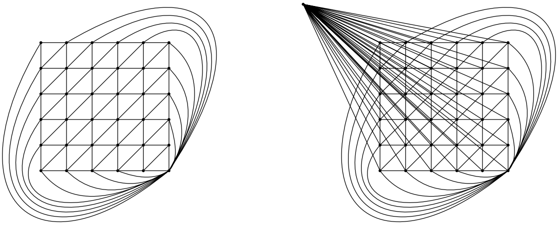


Figure: Γ_6 and Π_6

G' is a contraction of G if G' can be obtained by contracting edges of G

"Induced-wall theorem": proof 1 (when *tw* is large)

Let G be s.t. $tw(G) \geq f_H(k)$, then G contains Γ_k or Π_k

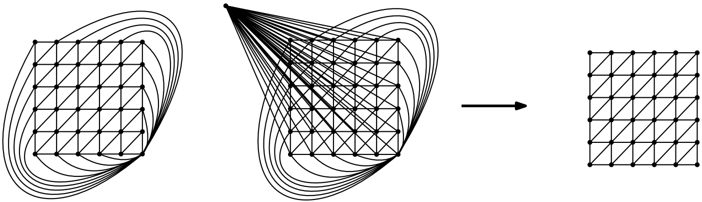


Figure: We can extract a *triangulated grid*

"Induced-wall theorem": proof 1 (when *tw* is large)

Let G be s.t. $tw(G) \geq f_H(k)$, then G contains Γ_k or Π_k

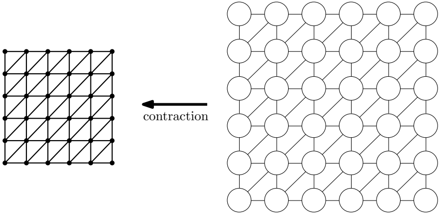


Figure: Consider the graph containing the contracted triangulated grid

"Induced-wall theorem": proof 1 (when *tw* is large)

Let G be s.t. $tw(G) \geq f_H(k)$, then G contains Γ_k or Π_k

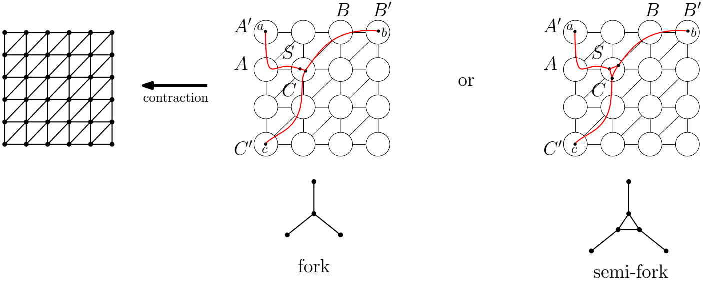


Figure: For some constant size of the triangulated grid, we find *forks* and *semiforks*

"Induced-wall theorem": proof 1 (when tw is large)

Let G be s.t. $tw(G) \geq f_H(k)$, then G contains Γ_k or Π_k

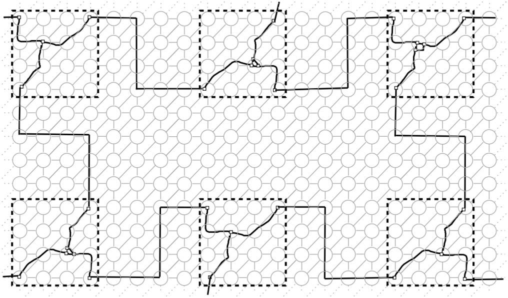


Figure: Combining them, we get a *large stone wall*

"Induced-wall theorem": proof 2 (cleaning stone wall)

Theorem

$\forall r \geq 2$ integer, $\exists\ n = n(r)$ integer s.t. every $(n \times n)$-stone wall contains an $(r \times r)$-wall or the $(r \times r)$-co-wall as induced subgraph.

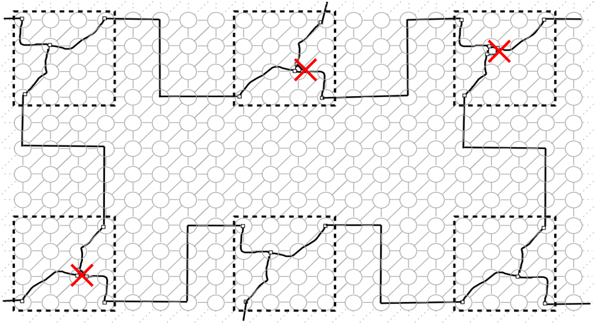


Figure: From an $(n \times n)$-stone wall W, contract each triangle into a vertex, we get wall W'

"Induced-wall theorem": proof 2 (cleaning stone wall)

Theorem

$\forall r \geq 2$ integer, $\exists\ n = n(r)$ integer s.t. every $(n \times n)$-stone wall contains an $(r \times r)$-wall or the $(r \times r)$-co-wall as induced subgraph.

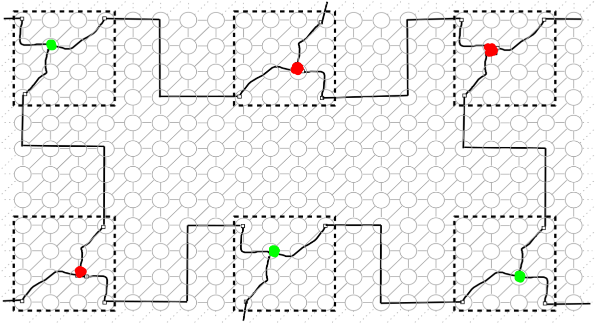
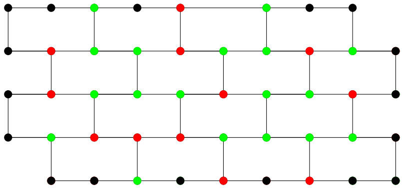


Figure: Color *the contracted vertex* with red and t*he other degree-3-vertices* with green

"Induced-wall theorem": proof 2 (cleaning stone wall)

Theorem

$\forall r \geq 2$ integer, $\exists\ n = n(r)$ integer s.t. every $(n \times n)$-stone wall contains an $(r \times r)$-wall or the $(r \times r)$-co-wall as induced subgraph.



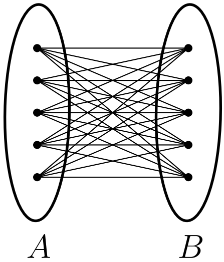
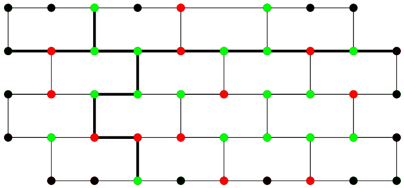


Figure: Build the *complete bipartite graphs* with partitions the *horizontal* paths and the *vertical* paths

"Induced-wall theorem": proof 2 (cleaning stone wall)

Theorem

$\forall r \geq 2$ integer, $\exists\ n = n(r)$ integer s.t. every $(n \times n)$-stone wall contains an $(r \times r)$-wall or the $(r \times r)$-co-wall as induced subgraph.



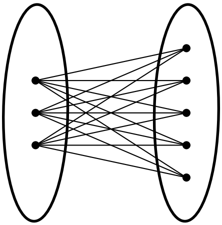
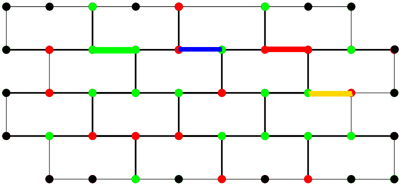


Figure: Each V-path and H-path intersect at two vertices, except the first and the last H-paths

"Induced-wall theorem": proof 2 (cleaning stone wall)

Theorem

$\forall r \geq 2$ integer, $\exists\ n = n(r)$ integer s.t. every $(n \times n)$-stone wall contains an $(r \times r)$-wall or the $(r \times r)$-co-wall as induced subgraph.



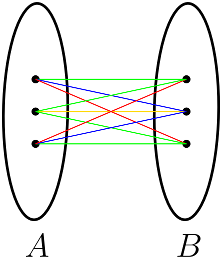
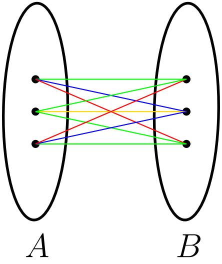


Figure: Edge-color the vertices of $K_{\cdot,\cdot}$ with red, green, blue, and yellow

"Induced-wall theorem": proof 2 (cleaning stone wall)



Lemma (Beineke and Schwenk, 1975)

$K_{\cdot,\cdot}$ contains a large monochromatic complete bipartite subgraph

1. GREEN: we obtain a *large wall*
2. RED: we obtain a *large co-wall* (undoing the contractions)
3. BLUE or YELLOW: we obtain a *large wall* (by local rerouting of the paths)

2nd part: even-hole-free graphs of bounded degree

Conjecture (Aboulker, Adler, Kim, S., Trotignon, 2020)

Even-hole-free graphs with bounded degree have bounded tree-width.

2nd part: even-hole-free graphs of bounded degree

Conjecture (Aboulker, Adler, Kim, S., Trotignon, 2020)

Even-hole-free graphs with bounded degree have bounded tree-width.

We prove the following cases:

- Subcubic even-hole-free graphs have $tw \leq 3$
- (Even hole, pyramid)-free graphs with $\Delta = 4$ have bounded tw

Structure theorem of subcubic even-hole-free graphs

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

Let G be a (theta, prism)-free subcubic graph. Then either:

- *G is a basic graph; or*
- *G has a clique separator of size at most 2; or*
- *G has a proper separator.*

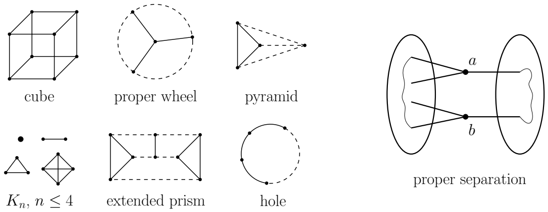


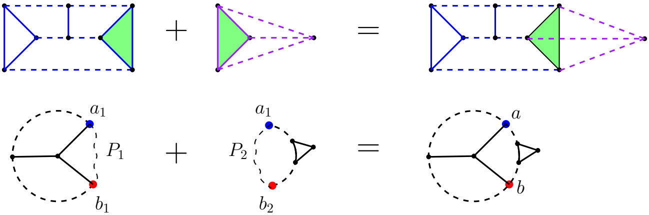
Figure: Basic graphs and proper separator

Tree-width of subcubic even-hole-free graphs

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)
Subcubic even-hole-free graphs have tree-width ≤ 3.

Sketch of proof.

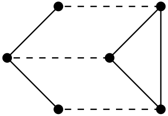
- Every basic graph has tree-width at most 3
- "Gluing" along a clique and proper gluing preserve tree-width



Tree-width of (even hole, pyramid)-free graphs with $\Delta = 4$

Theorem (S., Trotignon, 2020)

(Even hole, pyramid)-free graphs with $\Delta = 4$ *have tw* ≤ 4.

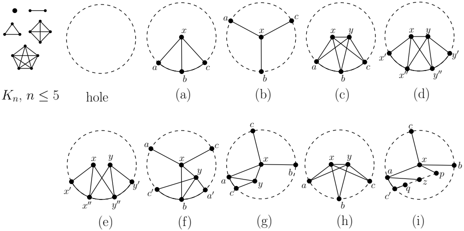


Tree-width of (even hole, pyramid)-free graphs with $\Delta = 4$

Keyproof:

If G is an (even hole, pyramid)-free graph with $\Delta(G) \leq 4$, then:

- G is a basic graph; or
- G has a clique separator of size at most 3; or
- G has a *p*roper separator for the class.



Even-hole-free graphs of bounded degree

The general case of bounded maximum degree is proven!

Theorem (Abrishami, Chudnovsky, Vušković, 2020)

Even-hole-free graphs of bounded degree have bounded tree-width. (This is actually proven for a superclass of ehf graphs.)

Approach: balanced separator + structural properties of the graphs

Open problems

Motivation: grid-minor theorem of Robertson and Seymour

There is a function f such that if $tw(G) > f(k)$, then G contains (as an induced subgraph) one of the following:

- a $(k \times k)$-wall or its subdivision
- line graph of a subdivision of a $(k \times k)$-wall
- a vertex of degree at least k

References

P. Aboulker, I. Adler, E. J. Kim, N. L. D. Sintiari, and N. Trotignon.
On the tree-width of even-hole-free graphs.
CoRR, abs/2008.05504, 2020.

T. Abrishami, M. Chudnovsky, and K. Vušković.
Even-hole-free graphs with bounded degree have bounded treewidth.
CoRR, abs/2009.01297, 2020.

N. L. D. Sintiari and N. Trotignon.
(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1 : Layered wheels.
CoRR, abs/1906.10998, 2019.

K. Vušković.
Even-hole-free graphs: a survey.
Applicable Analysis and Discrete Mathematics, 2010.

Thank you for listening!